Temporal multiscaling in hydrodynamic turbulence

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On the basis of the Navier-Stokes equations, we develop the high Reynolds number statistical theory of different-time, many-point spatial correlation functions of velocity differences. We find that their time dependence is *not* scale invariant: n-order correlation functions exhibit infinitely many distinct decorrelation times that are characterized by anomalous dynamical scaling exponents. We derive exact scaling relations that bridge all these dynamical exponents to the static anomalous exponents ζ_q of the standard structure functions. We propose a representation of the time dependence using the Legendre-transform formalism of multifractals that automatically reproduces all the newly found bridge relationships. [S1063-651X(97)12406-0]

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I. INTRODUCTION

Experimental investigations of the statistical objects that characterize the small-scale structure of turbulent flows are almost invariably based on a single-point measurement of the velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ as a function of time [1–4]. The Taylor ''frozen turbulence'' hypothesis is then used to surrogate time for space. The results of this type of measurements are ''simultaneous'' correlation of the velocity field itself or of velocity differences across a scale R (structure functions), or of velocity gradient fields like the dissipation field. A theoretical analysis which starts with the Navier-Stokes equations, on the other hand, states unequivocally that a closed-form theory for the simultaneous many space-point correlation functions of velocity differences is not available: attempting to derive equations for simultaneous correlation functions one finds integrals over time differences of space-time correlation and response functions [5]. There are no small parameters like a ratio of time scales (as in turbulent advection [6]) or a small interaction (like in weak turbulence [7]) that allow a reduction of such a theory to a closed scheme in terms of simultaneous objects only. In any theory that attempts to compute the scaling exponents of simultaneous correlation functions from first principles, one faces the computation of integrals over time of products of many-time correlation functions. It is important therefore to address different-time correlation functions for the sake of the fundamental theory of Navier-Stokes turbulence.

In addition, different time correlation functions appear in the theory of turbulent advection [8,9]. For example, the ''eddy diffusivity'' $h_{ij}(\boldsymbol{R})$ in the case of Gaussian, rapidly varying velocity field is a time integral over correlation function of the Lagrangian velocity difference $\boldsymbol{W}(\boldsymbol{r},\boldsymbol{r}',t_0|t)=\boldsymbol{V}(\boldsymbol{r}',t_0|t)-\boldsymbol{V}(\boldsymbol{r},t_0|t)$ [6]:

$$h_{ij}(\boldsymbol{r}-\boldsymbol{r}')\equiv\int_0^\infty d\tau\langle W_i(\boldsymbol{r},\boldsymbol{r}',0|\tau)W_j(\boldsymbol{r},\boldsymbol{r}',0|0)\rangle\ , \quad (1)$$

where $\langle\ \rangle$ denotes averaging over a space-homogeneous, stationary ensemble. The theory of advection by more realistic velocity fields calls for a knowledge of additional integrals over time of higher-order different-time many-point spatial correlation function of velocity differences.

The aims of this short paper are to initiate an analytic theory of such correlation functions, to find their scaling properties, to introduce their dynamical scaling exponents, and to relate them to objects that are known from standard experiments. Finally we will propose a representation of the time correlation functions in terms of the Legendre-transform formalism of multifractals. This representation will turn out to be of crucial importance for the evaluation of the scaling exponents in turbulence from first principles. We rely on the equations of fluid mechanics, without recourse to *ad hoc* models. In Sec. II we review briefly the equations of motion in the Belinicher-L'vov representation, and introduce and compute some typical time scales that are associated with n-point correlation functions. In Sec. III we show that these results show that the correlation functions are not scale invariant in their time argument. On the other hand, they exhibit temporal multiscaling for any non-Kolmogorov (i.e., anomalous) static scaling theory. Finally, in Sec. IV we offer a useful representation of the time correlation functions in terms of the Legendre-transform formalism, and point to the road ahead.

II. EQUATIONS OF MOTION AND DECORRELATION TIMES

In considering the decorrelation times of different-time, many-point spatial correlation functions, we need to make a choice of which velocity field we take as our fundamental field. The Eulerian velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ will not do, simply because its decorrelation time is dominated by the sweeping of small scales by large scale flows. In Ref. [5] we showed that at least from the point of view of the perturbative theory one can dispose of the sweeping effect using the Belinicher-L'vov (BL) velocity field $\boldsymbol{v}(\mathbf{r}_0,t_0|\boldsymbol{r},t)$ whose decorrelation time is intrinsic to the scale of consideration. In terms of the Eulerian velocity $\boldsymbol{u}(\boldsymbol{r},t)$ the BL velocity $\boldsymbol{v}(\mathbf{r}_0,t_0|\boldsymbol{r},t)$ was defined as [10]

$$\boldsymbol{v}(\boldsymbol{r}_0,t_0|\boldsymbol{r},t)\equiv\boldsymbol{u}[\boldsymbol{r}+\boldsymbol{\rho}_{\rm L}(\boldsymbol{r}_0,t_0|t),t]\ , \quad (2)$$

where $\boldsymbol{\rho}_{\rm L}(\boldsymbol{r}_0,t_0|t)$ is the Lagrangian trajectory of the fluid particle positioned at point $\boldsymbol{r}_0$ at time t_0. The observations of Belinicher and L'vov was that the variables $\boldsymbol{v}(\boldsymbol{r}_0,t_0|\boldsymbol{r},t)$ satisfy a Navier-Stokes-like equation in the limit of incompressible fluid, and that their simultaneous correlators are identical to the simultaneous correlators of $\boldsymbol{u}(\boldsymbol{r},t)$. In this sense these variables are more convenient than Lagrangian velocities $\boldsymbol{V}(\boldsymbol{r},t_0|t)\equiv\boldsymbol{u}[\boldsymbol{r}+\boldsymbol{\rho}_{\rm L}(\boldsymbol{r},t_0|t),t$ which do not satisfy a closed-form equation of motion.

Introduce now a difference of two (simultaneous) BL velocities at points $\boldsymbol{r}$ and $\boldsymbol{r}'$:

$$\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0,t_0|\boldsymbol{r},\boldsymbol{r}',t)\equiv\boldsymbol{v}(\boldsymbol{r}_0,t_0|\boldsymbol{r},t)-\boldsymbol{v}(\boldsymbol{r}_0,t_0|\boldsymbol{r}',t). \tag{3}$$

The equation of motion for $\boldsymbol{\mathcal{W}}$ can be calculated starting from the Navier-Stokes equation for the Eulerian field,

$$\left[\frac{\partial}{\partial t}+\hat{\mathcal{L}}-\nu(\nabla_r^2+\nabla_r'^2)\right]\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0,t_0|\boldsymbol{r},\boldsymbol{r}',t)=0. \tag{4}$$

We introduced an operator $\hat{\mathcal{L}}=\hat{\mathcal{L}}(\boldsymbol{r}_0,t_0|\boldsymbol{r},\boldsymbol{r}',t)$,

$$\hat{\mathcal{L}}(\boldsymbol{r}_0,t_0|\boldsymbol{r},\boldsymbol{r}',t)\equiv\overleftrightarrow{\boldsymbol{P}}\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0,t_0|\boldsymbol{r},\boldsymbol{r}_0,t)\cdot\nabla_r$$
$$+\overleftrightarrow{\boldsymbol{P}}'\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0,t_0|\boldsymbol{r}',\boldsymbol{r}_0,t)\cdot\nabla_r', \tag{5}$$

where $\overleftrightarrow{\boldsymbol{P}}$ is the usual transverse projection operator which is formally written as $\overleftrightarrow{\boldsymbol{P}}\equiv-\nabla^{-2}\nabla\times\nabla\times$. The application of $\overleftrightarrow{\boldsymbol{P}}$ to any given vector field $\boldsymbol{a}(\boldsymbol{r})$ is nonlocal, and it has the form

$$[\overleftrightarrow{\boldsymbol{P}}\boldsymbol{a}(\boldsymbol{r})]_\alpha=\int d\tilde{\boldsymbol{r}}P_{\alpha\beta}(\boldsymbol{r}-\tilde{\boldsymbol{r}})a_\beta(\tilde{\boldsymbol{r}}). \tag{6}$$

The explicit form of the kernel can be found, for example, in Ref. [1]. In Eq. (5), $\overleftrightarrow{\boldsymbol{P}}$ and $\overleftrightarrow{\boldsymbol{P}}'$ are projection operators which act on fields that depend on the corresponding coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$. The equation of motion (4) forms the basis of the following discussion of the time correlation functions.

The fundamental statistical quantities in our study are the different-time, many-point, ''fully unfused,'' n-rank-tensor correlation function of the BL velocity differences $\boldsymbol{\mathcal{W}}_j\equiv\boldsymbol{\mathcal{W}}(\boldsymbol{r}_0,t_0|\boldsymbol{r}_j\boldsymbol{r}_j',t_j)$:

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{r}_0,t_0|\boldsymbol{r}_1\boldsymbol{r}_1',t_1\cdots\boldsymbol{r}_n\boldsymbol{r}_n',t_n)=\langle\boldsymbol{\mathcal{W}}_1\cdots\boldsymbol{\mathcal{W}}_n\rangle. \tag{7}$$

A. Two-time quantities

We begin the development with the simplest nonsimultaneous case in which there are two different times in Eq. (7). Choose $t_i=t+\tau$ for every $i\leqslant p$ and $t_i=t$ for every $i>p$. We will denote the correlation function with this choice of times as $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$, denoting for brevity the rest of the separations as $\{\boldsymbol{R}_j\}$. The τ derivative of $\boldsymbol{\mathcal{F}}_{n,p}$ is

$$\frac{\partial\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)}{d\tau}=\sum_{j=1}^{p}\left\langle\boldsymbol{\mathcal{W}}_1\cdots\frac{\partial\boldsymbol{\mathcal{W}}_j}{\partial t}\cdots\boldsymbol{\mathcal{W}}_n\right\rangle.$$

Using the equation of motion (4), we find

$$\frac{\partial\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)}{d\tau}+\boldsymbol{\mathcal{D}}_{n,p}(\{\boldsymbol{R}_j\},\tau)=\boldsymbol{\mathcal{J}}_{n,p}(\{\boldsymbol{R}_j\},\tau),$$

$$\boldsymbol{\mathcal{D}}_{n,p}(\{\boldsymbol{R}_j\},\tau)=\sum_{j=1}^{p}\langle\boldsymbol{\mathcal{W}}_1\cdots\hat{\mathcal{L}}_j\boldsymbol{\mathcal{W}}_j\cdots\boldsymbol{\mathcal{W}}_n\rangle, \tag{8}$$

$$\boldsymbol{\mathcal{J}}_{n,p}(\{\boldsymbol{R}_j\},\tau)=\nu\sum_{j=1}^{p}(\nabla_j^2+\nabla_j'^2)\langle\boldsymbol{\mathcal{W}}_1\cdots\boldsymbol{\mathcal{W}}_j\cdots\boldsymbol{\mathcal{W}}_n\rangle,$$

with $\hat{\mathcal{L}}_j\equiv\hat{\mathcal{L}}(\boldsymbol{r}_0,t_0|\boldsymbol{r}_j,\boldsymbol{r}_j',t)$. We remember that $\hat{\mathcal{L}}_j\boldsymbol{\mathcal{W}}_j$ is a nonlocal object that is quadratic in BL velocity differences, cf. Eq. (5).

To understand the role of the various contributions in Eq. (8), we first note that in the limit $\nu\to0$ the term $\boldsymbol{\mathcal{J}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ vanishes. To see this, note that in the fully unfused case this term is bounded from above by $C\nu\boldsymbol{\mathcal{J}}_{n,p}(\{\boldsymbol{R}_j\},0)/R_{\rm min}^2$, where C is a ν-independent constant and $R_{\rm min}$ is the minimal separation between the coordinates. There is nothing in this quantity to balance ν in the limit $\nu\to0$. This is of course the advantage of working with fully unfused quantities; we could not do this with the balance equation for fused correlators [say structure functions $S_n(R)$] in which the dissipative term reaches a finite limit when $\nu\to0$. Thus for $\nu\to0$, or for very large Reynolds numbers, we have

$$\partial\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)/\partial\tau+\boldsymbol{\mathcal{D}}_{n,p}(\{\boldsymbol{R}_j\},\tau)=0. \tag{9}$$

Next we note that the term $\boldsymbol{\mathcal{D}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ contains an integration over all space because of the projection operator Eq. (6). The analysis of this integral calls for the use of ''fusion rules'' [11] that determine the asymptotic properties of many-point spatial correlation functions when a group of coordinates fuse together. When the dummy integration variable $\tilde{\boldsymbol{r}}$ comes close to one of the coordinates in the correlation function, we use the fusion rules of correlation functions with one pair of coalescing coordinates, and when $\tilde{\boldsymbol{r}}\to\infty$ all the other coordinates are coalesced together compared to this coordinate. Knowing the asymptotic properties via the fusion rules, we can prove a very important result, i.e., that the integral in $\boldsymbol{\mathcal{D}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ which originates from the projection operator converges in the infrared and ultraviolet regimes. The proof of convergence was discussed in Ref. [12], but due to its importance for the present development we reproduce it in Appendix A. Here we draw the important conclusions: the convergence of the integral means that when all the separations $\boldsymbol{R}_j\equiv\boldsymbol{r}_j-\boldsymbol{r}_j'$ are of the same order R, the largest contribution to the integral comes from the region where $\tilde{\boldsymbol{r}}$ is separated by R from all the coordinates. In other words, since the integral converges for $\tilde{r}\ll R$ as well as for $\tilde{r}\gg R$, and since the integrand is a scale invariant function of its argument (having no characteristic scale besides R) we can evaluate it as some R-independent constant times the integral obtained from putting $\tilde{r}\approx R$ in the integrand. When this is done the integrand can be evaluated as $\boldsymbol{\mathcal{F}}_{n+1,p+1}(\{\boldsymbol{R}_j\},\tau)/R$. As a result one can easily prove that there exist two constants C_1 and C_2 such that

$$C_1\boldsymbol{\mathcal{F}}_{n+1,p+1}/R\leqslant\boldsymbol{\mathcal{D}}_{n,p}(\tau)\leqslant C_2\boldsymbol{\mathcal{F}}_{n+1,p+1}/R. \tag{10}$$

Scaling wise, $\mathcal{D}_{n,p}(\{\boldsymbol{R}_j\},\tau)\sim\mathcal{F}_{n+1,p+1}(\{\boldsymbol{R}_j\},\tau)/R$. We will show now that this result has far-reaching consequences for the dynamical exponents.

Introduce the typical decorrelation time ${}^1\tau_{n,p}(R)$ that is associated with the one-time difference quantity $\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ when all R_j are of the order of R:

$$^1\tau_{n,p}(R)\equiv\int_0^\infty d\tau\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},\tau)/\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},0). \tag{11}$$

Integrate Eq. (9) in the interval $(0,\infty)$, use evaluation (10), and derive

$$^1\tau_{n+1,p+1}(R)\sim R\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},0)/\mathcal{F}_{n+1,p+1}(\{\boldsymbol{R}_j\},0). \tag{12}$$

In this equation, $\tau=0$, and we have the simultaneous correlation functions $\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},0)$ of BL velocity differences which *are identical* [5] to the corresponding simultaneous correlation functions of Eulerian velocity differences, and therefore $\mathcal{F}_{n,p}(\{\boldsymbol{R}_j\},0)\sim S_n(R)$. We see that from the point of view of scaling there is no p dependence in this equation: for different values of p, only the coefficients can change. We thus can use the value

$$^1\tau_n(R)\sim RS_{n-1}(R)/S_n(R)\propto R^{z_{n,1}} \tag{13}$$

for the estimate of the decorrelation time ${}^1\tau_{n,p}(R)$ of a one-time-difference correlation function for any value of p. Here we introduced the dynamical scaling exponent $z_{n,1}$ that characterizes this time scale. In terms of the scaling exponents of the structure functions, $S_n(R)\sim R^{\zeta_n}$, we can write the ''bridge relations''

$$z_{n,1}=1+\zeta_{n-1}-\zeta_n. \tag{14}$$

These are the first results of this paper.

B. Three and more time quantities

Next we ask whether the same time scale also characterizes correlation functions having two or more time separations. To this aim consider next the three-time quantity that is obtained from $\mathcal{F}_n$ by choosing $t_i=t+\tau_1$ for $i\leqslant p$, $t_i=t+\tau_2$ for $p<i\leqslant p+q$, and $t_i=t$ for $i>p+q$. We denote this quantity as $\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)$. We define the decorrelation time ${}^2\tau_{n,p,q}$ of this quantity by

$$^2\tau_{n,p,q}(R)\equiv\left(\frac{\int_0^\infty d\tau_1 d\tau_2\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)}{\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},0,0)}\right)^{1/2}. \tag{15}$$

One could think naively that this decorrelation time is of the same order as ${}^1\tau_n(R)$; see Eq. (13). The calculation leads to a different result. To see this calculate the double derivative of $\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)$ with respect to τ_1 and τ_2. We begin with definition (7), and compute directly the two time derivatives with respect to τ_1 and τ_2. This results in a new balance equation. For the fully unfused situation, and in the limit $\nu\to0$, we find

$$\partial^2\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)/\partial\tau_1\partial\tau_2+\mathcal{D}_{n,p,q}=0, \tag{16}$$

where now $\mathcal{D}_{n,p,q}=\mathcal{D}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)$, and

$$\mathcal{D}_{n,2}^{(p,q)}=\sum_{j=1}^{p}\sum_{k=p+1}^{p+q}\langle\mathcal{W}_1\cdots\hat{\mathcal{L}}_j\mathcal{W}_j\cdots\hat{\mathcal{L}}_k\mathcal{W}_k\cdots\mathcal{W}_n\rangle.$$

On the right-hand side of Eq. (16), we neglected two terms that vanish in the limit $\nu\to0$. The expression for $\mathcal{D}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)$ contains two space integrals that originate from the two projection operators which are hidden in $\hat{\mathcal{L}}_j$ and $\hat{\mathcal{L}}_k$. Using the same ideas when all the separations are of the order of R we can estimate with impunity

$$\mathcal{D}_{n,p,q}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)\sim\mathcal{F}_{n+2,p+1,q+1}(\{\boldsymbol{R}_j\},\tau_1,\tau_2)/R^2. \tag{17}$$

Now integrating Eq. (16) over τ_1 and τ_2 in the interval $[-\infty,0]$, and remembering that the simultaneous correlation functions of the BL and Eulerian velocities coincide [and therefore $\mathcal{F}_{n,p,q}(\{\boldsymbol{R}_j\},0,0)\sim S_n(R)$], we find

$$^2\tau_{n+2,p+1,q+1}(R)\sim R\sqrt{S_n(R)/S_{n+2}(R)}\propto R^{z_{n+2,2}}. \tag{18}$$

As before, the scaling exponent of ${}^2\tau_{n,p,q}(R)$ are independent of p and q, and we find that

$$z_{n,2}=1+(\zeta_{n-2}-\zeta_n)/2 \tag{19}$$

which is different from Eq. (14). We see that the naive expectation is not realized. The difference between the two scaling exponents $z_{n,1}-z_{n,2}=\zeta_{n-1}-(\zeta_n+\zeta_{n-2})/2$ is zero only for linear scaling, meaning that in that case the naive expectation that the time scales are identical is correct. On the other hand, for the situation of multiscaling the Hoelder inequalities require the difference to be positive. Accordingly, for $R\ll L$, we have $\tau_{n,2}(R)\gg\tau_{n,1}(R)$.

We can proceed with correlation functions $\mathcal{F}_n$ that depend on m time differences. Denoting this as $\mathcal{F}_{n,p_1,p_2\cdots p_m}(\{\boldsymbol{R}_j\},\tau_1\cdots\tau_m)$ we establish the exact scaling law for its decorrelation time ${}^m\tau_{n,p_1,p_2\cdots p_m}(R)$. Integrating the correlation functions over all its m different-time arguments, and normalizing by the simulataneous object, we obtain an estimate for the mth power of the decorrelation time. Acoordingly we can define the decorrelation time ${}^m\tau_{n,p_1,p_2\cdots p_m}(R)$ as

$$\left(\frac{\int_0^\infty d\tau_1\cdots d\tau_m\mathcal{F}_{n,p_1,p_2\cdots p_m}(\{\boldsymbol{R}_j\},\tau_1\cdots\tau_m)}{\mathcal{F}_{n,p_1,p_2\cdots p_m}(\{\boldsymbol{R}_j\},0,0\cdots0)}\right)^{1/m}. \tag{20}$$

Repeating the steps described above, we need to take m time derivative from definition (7), and obtain the analog of Eq. (16). Integrating over m times, we will find the analog of Eq. (17), but the index of the correlation function on the right-hand side will be raised by m, to $n+m$. Accordingly, we find that the dynamical scaling exponent of the $m+n$th-order correlation function scales like $R(S_n(R)/S_{n+m}(R))^{1/m}$. In terms of the dynamical exponent $z_{n,m}$ of the nth-order correlation function that characterizes $\tau_{n,m}$ (when all the separations are of the order of R), and $\tau_{n,m}\propto R^{z_{n,m}}$, we find

$$z_{n,m}=1+(\zeta_{n-m}-\zeta_n)/m\ ,\quad n-m\leqslant2\ . \tag{21}$$

These are also results of this paper, generalizing Eq. (14). One can see, using the Hoelder inequalities, that $z_{n,m}$ is a nonincreasing function of m for fixed n, and in a multiscaling situation they are decreasing. The meaning is that the larger m is the *longer* is the decorrelation time of the corresponding $m+1$ time correlation function, $\tau_{n,p}(R) \gg \tau_{n,q}(R)$ for $p<q$. We conclude therefore with a set of scaling relations (21) that indicate that, on the one hand, the dynamical scaling properties are very nontrivial, and, on the other hand, all the dynamical scaling exponents can be computed from the knowledge of the scaling exponents ζ_n of the structure functions. The latter result follows from the fusion rules that render the integrals in the interaction terms local.

III. BREAKDOWN OF ''DYNAMICAL SCALING''

The fundamental idea of temporal scale invariance is that a quantity like $\mathcal{F}_{n,p}(\{\boldsymbol{R}\},t)$ is characterized by a *dynamical* scaling exponent z, such that

$$\mathcal{F}_{n,p}(\{\lambda\boldsymbol{R}\},\lambda^z t)=\lambda^{\zeta_n}\mathcal{F}_{n,p}(\{\boldsymbol{R}\},t), \tag{22}$$

where ζ_n is the standard ''static'' scaling exponent. Results (21) indicate that our correlation functions do not exhibit such temporal scale invariance. To see this in another way, we consider higher-order temporal moments of the one-time-difference correlation functions $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ defined by

$$\mathcal{T}^{(k)}_{n,p}(R)\equiv\frac{\int_0^\infty d\tau\ \tau^{k-1}\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)}{\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},0)}. \tag{23}$$

The intuitive meaning of $\mathcal{T}^{(k)}_{n,p}(R)$ is a *k-order decorrelation moment* of $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ whose dimension is $(\text{time})^k$. The first-order decorrelation moment $\mathcal{T}^{(1)}_{n,p}(R)$ is the previously defined decorrelation time ${}^1\tau_{n,p}$. To find the scaling exponents of these quantities we start with Eq. (9), multiply by τ^k, and integrate over τ in the interval $(0,\infty)$. Using evaluation (10) and assuming convergence of the integrals over τ, we derive

$$-k\int_0^\infty \boldsymbol{\mathcal{F}}_{n,p}\tau^{k-1}d\tau\sim\frac{1}{R}\int_0^\infty \boldsymbol{\mathcal{F}}_{n+1,p+1}\tau^k d\tau, \tag{24}$$

where we have integrated by parts on the left-hand side. We stress that in deriving this equation we assert that $k+1$ moments exist; this is not known *a priori*. Using definition (23), for all the separations of the order of R we find the recurrence relation

$$RS_n(R)\mathcal{T}^{(k)}_{n,p}\sim S_{n+1}(R)\mathcal{T}^{(k+1)}_{n+1,p+1}. \tag{25}$$

The solution (up to the p-dependent coefficient) is

$$\mathcal{T}^{(k)}_{n,p}(R)\sim\mathcal{T}^{(k)}_{n}(R)=R^kS_{n-k}(R)/S_n(R) \tag{26}$$

for $k\leq n-2$. The procedure does not yield information about higher k values. Thus, for the scaling exponents of the decorrelation moments we find

$$\mathcal{T}^{(k)}_{n}(R)\propto R^{z^{(k)}_n},\ \ z^{(k)}_n=k+\zeta_{n-k}-\zeta_n=kz_{n,k}. \tag{27}$$

For a scale invariant function like Eq. (22) we expect to find that $z^{(k)}_n/k$ is k independent, as can be easily seen from substituting Eq. (22) into Eq. (23). Clearly, relation (27) is k independent only in the case of linear scaling, but any nonlinear dependence of ζ_n on n ruins the k independence. We learn from an analysis of the moments that there is no single typical time which characterizes the τ dependence of $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}\},\tau)$. There is no simple ''dynamical scaling exponent'' z that can be used to collapse the time dependence in the form $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}\},\tau)\sim R^{\zeta_n}f(\tau/R^z)$. *Even the two-time correlation function is not a scale-invariant object*. In this respect it is similar to the probability distribution function of the velocity differences across a scale R, for which the spectrum of ζ_n is a reflection of the lack of scale invariance.

IV. TEMPORAL MULTISCALING REPRESENTATION

This section does not introduce new results, but offers a convenient presentation of the time dependence of the correlation functions. For concreteness we restrict the description to the n-point two-time correlation function $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau)$, and will further assume that all the separations R_j are of the same order R. Consider first the simultaneous function $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}_j\},\tau=0)$. Following the standard ideas of multifractals [2,14] the simultaneous function can be represented as

$$\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}\},\tau=0)\sim U^n\int_{h_{\min}}^{h_{\max}}d\mu(h)\left(\frac{R}{L}\right)^{\mathcal{Z}(n,h)}. \tag{28}$$

Here U is a typical velocity scale, and the function $\mathcal{Z}(n,h)$ is defined as

$$\mathcal{Z}(n,h)\equiv nh+3-\mathcal{D}(h). \tag{29}$$

The function $\mathcal{D}(h)$ is related to the scaling exponents ζ_n via the usual Legendre transformation

$$\zeta_n=\min_h\mathcal{Z}(n,h). \tag{30}$$

As usual, the integral in Eq. (28) is computed in the limit $R/L\to0$ via the steepest descent method. Neglecting logarithmic corrections, one finds that $\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}\},0)\propto R^{\zeta_n}$.

The physical intuition behind representation (28) is that there are velocity field configurations that are characterized by different scaling exponents h. For different orders n the main contribution comes from that value of h that determines the position of the saddle point in integral (28). This intuition is extended to the time domain. The particular velocity configurations that are characterized by an exponent h also display a typical time scale $\tau_{R,h}$ which is written as

$$\tau_{R,h}\approx\frac{R}{U}\left(\frac{L}{R}\right)^h. \tag{31}$$

Accordingly we propose a new temporal multiscaling representation for the time-dependent function,

$$\boldsymbol{\mathcal{F}}_{n,p}(\{\boldsymbol{R}\},\tau)\sim U^n\int_{h_{\min}}^{h_{\max}}d\mu(h)\left(\frac{R}{L}\right)^{\mathcal{Z}(n,h)}f^{(p)}_n\left(\frac{\tau}{\tau_{R,h}}\right). \tag{32}$$

The scaling functions $f_n^{(p)}(\tau/\tau_{R,h})$ are chosen with the boundary condition $f_n^{(p)}(0)=1$, and such that the Melin transform $m_n^{(p)}(\xi)$ exists,

$$m_n^{(p)}(\xi)\equiv\int_0^\infty \hat{\tau}^{\xi-1}f_n^{(p)}(\hat{\tau})d\hat{\tau}, \tag{33}$$

where $\hat{\tau}$ is a dimensionless variable.

To see that this representation reproduces the scaling relations (27), all that we need is to compute the Melin transform of $\mathcal{F}_{n,p}$:

$$M_m^{(p)}(\{\boldsymbol{R}\},\xi)\equiv\int_0^\infty \hat{\tau}^{\xi-1}\mathcal{F}_{n,p}(\{\boldsymbol{R}\},\hat{\tau}R/U)d\hat{\tau}, \tag{34}$$

where here the dimensionless dummy variable is chosen as $\hat{\tau}\equiv\tau U/R$. Substituting Eq. (32) into Eq. (34), computing the integral over $\hat{\tau}$ first, and performing the saddle point integral of $\mu(h)$, we find that

$$M_m^{(p)}(\{\boldsymbol{R}\},\xi)\propto R^{\xi+\zeta_{n-\xi}}. \tag{35}$$

For the case $\xi=k$ we recover, comparing definition (23) with Eq. (34), the scaling relations (27).

V. CONCLUSIONS

The main conclusion of this paper is that the standard "dynamical scaling" assumption fails for hydrodynamic turbulence. Instead, we have temporal multiscaling which can be represented in terms of infinitely many dynamical scaling exponents for every n-order correlation function. All these exponents are expressed in terms of the scaling exponents ζ_n of the standard structure functions $S_n(R)$. In a succinct way the temporal multiscaling is represented by Eq. (32).

Another simple way to remember all the scaling relations that were obtained above is given by the following simple rule. To obtain the dynamical scaling exponent, every integral over τ in the definition of the decorrelation time (20), and every factor τ in the definition of moments (23) can be traded for a factor of $R/\mathcal{W}$ *within the average* of the correlation function involved. The dynamical exponent is determined by the resulting scaling exponents of the resulting simultaneous correlation function.

The deep reason behind the temporal multiscaling and this simple rule is the nonperturbative locality (convergence) of the integrals appearing in $\mathcal{D}_n$. Because of this locality one can estimate from the equation of motion (4) $1/\tau\sim\partial/\partial\tau\sim\boldsymbol{\mathcal{W}}\cdot\nabla\sim\mathcal{W}(R)/R$. This means that we can use the substitutions $\tau,\int d\tau\Rightarrow R/\mathcal{W}(R)$ *as long as we use them within the average*, and when all the separations are of the order of R. We propose to refer to this substitution rule as "weak dynamical similarity," where "weak" is a reminder that the rules can be used *only* under the averaging procedure, and *only* for scaling purposes. The same property of locality of the interaction integrals was shown [13,12] to yield another set of bridge relations between scaling exponents of correlation functions of gradient fields and the scaling exponents ζ_n. Those relations can be summarized by another substitution rule that we refer to as "weak dissipative similarity;" it follows from equating the viscous and nonlinear terms in the equations of motion: $\nu\nabla^2\Rightarrow\mathcal{W}(R)/R$. Again "weak" refers to the reminder that we are only allowed to use these substitutions for scaling purposes within the average. Note that our weak dissipative similarity rule is weaker than the Kolmogorov refined similarity hypothesis which states the *dynamical* relationship $(\nabla\mathcal{W}(R))^2\sim\mathcal{W}^3/R$. Note that both our rules are derived from first principles.

Finally, we should ask whether the results presented above are particular to the time correlation function of BL velocity differences, or do they reflect intrinsic scaling properties that are shared by other dynamical presentations like the standard Lagrangian velocity fields. The answer is that the results are general; all that we have used are the property of convergence of the interaction integrals, and the fact that the simultaneous correlation functions of the BL fields are the same as those of the Eulerian velocities. These properties also hold for Lagrangian velocities, and in fact for any sensible choice of velocity representation in which the sweeping effect is eliminated. Accordingly we state that the dynamical exponent are invariant to the representation and in particular will be the same for many-time correlation functions of Lagrangian velocity differences.

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APPENDIX: LOCALITY OF THE INTERACTION TERM

To prove the convergence of the interaction term in Eq. (9), we note that it is sufficient to prove the convergence of the integral when the the correlation function $\mathcal{D}_{n,p}(\{\boldsymbol{R}_j\},\tau)$ is replaced in the integrand by the simultaneous correlation function $\mathcal{D}_n(\{\boldsymbol{R}_j\})$. The reason is that the latter is always larger or equal to the former, so the convergence of the integral over the simultanous correlation function guarantees the convergence of the integral over the time corrlation function. We remember that we deal with the most general configuration of coordinates in which all the $2n$ coordinates $\boldsymbol{r}_j$ and $\boldsymbol{r}_j'$ are different, but we specialize the discussion to the case in which all the $n(2n-1)$ separations are of the same order of magnitude, which we designate by R. We will show that the integral over $\tilde{\boldsymbol{r}}$ appearing in Eq. (9) but with $\mathcal{D}_n$ as the integrand is "local" in the following sense. First it converges in the "ultraviolet" limit. This limit has to be considered when (i) $\tilde{r}\to 0$, (ii) when $(\boldsymbol{r}_j-\tilde{\boldsymbol{r}})$ becomes very close to any of the $2n-1$ coordinates other than r_j, and (iii) when $(\boldsymbol{r}_j'-\boldsymbol{r})$ becomes very close to any of the $2n-1$ coordinates other than r_j'. Second, it converges in the "infrared" limit when $\tilde{r}\to\infty$. The idea for the proof of these properties lies in the use of the fusion rules which were presened in detail in Refs. [11,12].

1. Ultraviolet convergence

To demonstrate the convergence of the integral in $\mathcal{D}_n$ in the ultraviolet region we can consider any term from the sum on j. Writing $u_\gamma(\boldsymbol{r}_j-\boldsymbol{r})=(\Sigma_{k=1}^n u_\gamma(\boldsymbol{r}_j-\boldsymbol{r}))/n$, we write one of the k terms in the sum. The integral that appears is of the form

$$I=\frac{1}{n}\int d\boldsymbol{r}P_{\alpha_j\beta}(\boldsymbol{r})\frac{\partial}{\partial r_{j\gamma}}\langle w_{\alpha_1}(\boldsymbol{r}_1,\boldsymbol{r}_1')\cdots w_\gamma(\boldsymbol{r}_j-\boldsymbol{r},\boldsymbol{r}_k)\times w_\beta(\boldsymbol{r}_j-\boldsymbol{r},\boldsymbol{r}_j'-\boldsymbol{r})\cdots w_{\alpha_n}(\boldsymbol{r}_n,\boldsymbol{r}_n')\rangle \quad . \tag{A1}$$

As the coordinate $\boldsymbol{r}$ is being integrated over, the most dangerous ultraviolet contribution comes from the region of small r. In this region the projection operator can be evaluated as $1/r^3$. Other coalescence events of $\boldsymbol{r}$ with other coordinates contribute less divergent integrands since the projection operator does not become singular. When r becomes small, there are two possibilities: (i) $\boldsymbol{r}_j\neq\boldsymbol{r}_k$ and (ii) $\boldsymbol{r}_j=\boldsymbol{r}_k$. In the first case the correlation function itself is analytic in the region $r\to 0$, and we can expand it in a Taylor series const$+\boldsymbol{B}\cdot\boldsymbol{r}+\cdots$, where $\boldsymbol{B}$ is an $\boldsymbol{r}$-independent vector. The constant term is annihilated by the projection operator. The term linear in $\boldsymbol{r}$ vanishes under the $d\boldsymbol{r}$ integration due to $\boldsymbol{r}\to-\boldsymbol{r}$ symmetry. The next term which is proportional to r^2 is convergent in the ultraviolet. In the second case we have a velocity difference across the length r. Accordingly we need to use the fusion rule [11,12], and learn that the leading contribution is proportional to r^{ζ_2}. This is not sufficient for convergence in the ultraviolet if the derivative with respect to r_j could be evaluated as $1/r$ when $\boldsymbol{r}_j=\boldsymbol{r}_k$. However, if we take into account the tensor structure of $\widetilde{F}_2^{\alpha\beta}$, we see that this dangerous contribution vanishes. Accordingly, the derivative is evaluated as the inverse of the distance between $\boldsymbol{r}_j$ and the nearest coordinate in the correlation function, leading to convergence in the ultraviolet.

2. Infrared convergence

To understand the convergence of $\mathcal{D}_n$ when the integration variable r becomes very large we can consider again the typical term (A1). The relevant geometry is shown in Fig. 1. There is one velocity difference across the coordinates $\boldsymbol{r}_j-\boldsymbol{r}$ and $\boldsymbol{r}_j'-\boldsymbol{r}$ (which is shown on the right of the figure), $n-1$ velocity differences across coordinates that are all within a ball of radius R (at the left of the figure), and one velocity difference across the large distance r which is much larger than R. In the notation of this figure the leading order contribution for large r is obtained from the fusion rules [11,12] with the following evaluation for the leading term:

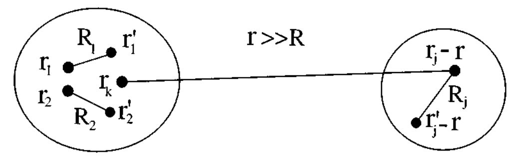

FIG. 1. Typical geometry with $n-1$ velocity differences in a ball of radius R on the left separated by a large distance $r\gg R$ from a pair of points on the right.

$$I\propto r^{\zeta_{n+1}}\left(\frac{R_j}{r}\right)^{\zeta_2}\left(\frac{R}{r}\right)^{\zeta_{n-1}} \quad . \tag{A2}$$

On the face of it, this term is near dangerous. For $K41$ scaling the r dependence cancels, and the integral is logarithmically divergent. For anomalous scaling the integral converges since $\zeta_{n+1}\leq\zeta_{n-1}+\zeta_2$ due to Hoelder inequalities. This convergence seems slow. However, the situation is in fact much safer. If we take into account the precise form of the second-order structure function in the fusion rules we find that the divergence with respect to $\boldsymbol{r}_j$ translates in fact to $\partial S_2^{\beta\gamma}(\boldsymbol{R}_j)/\partial R_{j\gamma}$ which is zero due to incompressibility. The next-order term is convergent even for simple ($K41$) scaling. This completes the proof of locality of the interaction term in Eq. (9).

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